Response to Hartry Field's 'Naive Truth Theory'

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Birkbeck, London

Plurals, Predicates and Paradox

Truth Conference Amsterdam 2011

1. Field gave a new argument against Excluded Middle.
2. He explained shortcomings of the logic proposed in his publications and suggested an alternative: $Ł_{\aleph_0} - (U)$.

One Challenge and One Suggestion

1. Today's *absurdity principles* conflict with Field's probabilistic understanding of rejection.
2. We know more about $Ł_{\aleph_0} - (U)$ than suggested in the talk.

Outline

Field's Argument from Rejection

- Field: for *consistent naive* truth, restrict Excluded Middle.
- He has argued from four principles of *rejection*.

R1 Reject$(\phi \wedge \neg T\ulcorner\phi\urcorner)$

R2 Reject$(\neg\phi \wedge T\ulcorner\phi\urcorner)$

R3 Reject$(\phi) \wedge$Reject$(\psi) \rightarrow$ Reject$(\phi \vee \psi)$

R4 Reject$(\phi \wedge \psi) \wedge (\phi \leftrightarrow \psi) \rightarrow$ Reject$(\phi) \wedge$Reject(ψ)

1.	$\lambda \leftrightarrow \neg T\ulcorner\lambda\urcorner$	
2.	Reject$(\lambda) \wedge$Reject$(\neg T\ulcorner\lambda\urcorner)$	R1 + R4
3.	Reject$(\lambda \vee \neg T\ulcorner\lambda\urcorner)$	R3
4.	Reject$(\lambda \vee \neg\lambda)$	Naivety of 'T'

- '... we should reject the application of the law of the excluded middle to the Liar sentence' (slide 7)

… doesn't square well with Classical Probability …

- Rejecting ϕ is to have a *low degree of belief* in ϕ [Field, 2005, p. 26][Field, 2008, p. 74].
  - Accept(ϕ) $:\leftrightarrow$ $P(\phi) > \tau$
  - Reject(ϕ) $:\leftrightarrow$ $P(\phi) < (1 - \tau)$
- If $P(\phi) + P(\neg\phi) = 1$ then Reject(ϕ) iff Accept$(\neg\phi)$.
- Accept(λ) iff Accept$(\neg T\ulcorner\lambda\urcorner)$ iff Reject$(T\ulcorner\lambda\urcorner)$ iff Reject(λ).
- Reject$(\lambda \vee \neg\lambda)$ iff Accept$\neg(\lambda \vee \neg\lambda)$ iff Accept$(\neg\lambda \wedge \lambda)$ iff As(λ) iff Accept$(\lambda \vee \neg\lambda)$

. . . nor with any other

- Field has noticed this [Field, 2005, p. 26], [Field, 2008, pp. 74n]
- His response: restrict classical probability theory.
- $P(\lambda) + P(\neg\lambda) < 1$

$P(\lambda) + P(\neg\lambda) > \tau$	Accept$(\lambda \vee \neg\lambda)$
$P(\lambda) + P(\neg\lambda) < (1 - \tau)$	Reject$(\lambda \vee \neg\lambda)$
$(1 - \tau) < P(\lambda) + P(\neg\lambda) < \tau$	Neither

- The degree-of-belief account of rejection either begs the question against Excluded Middle, or refutes his argument.

Suggestion

- Don't *reject* LEM, keep silent.
- $(1-\tau) < P(\lambda \vee \neg\lambda) < \tau$

  S1 $\text{Silent}(\phi \wedge \neg T\ulcorner\phi\urcorner)$

  S3 $\text{Silent}(\phi) \wedge \text{Silent}(\psi) \rightarrow \text{Silent}(\phi \vee \psi)$

  S4 $\text{Silent}(\phi \wedge \psi) \wedge (\phi \leftrightarrow \psi) \rightarrow \text{Silent}(\phi) \wedge \text{Silent}(\psi)$

  $\Rightarrow$ $\text{Silent}(\lambda \vee \neg\lambda)$
- 'Paracompleteness runs deep' [Field, 2008, p. 72]

Outline

The Offer from Field's Book

- Field constructs a revision sequence of Kripke fixed point models.
- This allows him to define a stronger conditional $\rightarrow$.
  - $\phi \rightarrow \phi$
  - $\phi \leftrightarrow T\ulcorner\phi\urcorner$
- *determinately* ϕ iff $\phi \wedge \neg(\phi \rightarrow \neg\phi)$
  - Any $\lambda^\alpha \leftrightarrow \neg \underbrace{DD\ldots D}_{\alpha\times}\lambda^\alpha$ is of value $\frac{1}{2}$ but $v(\neg D \underbrace{DD\ldots D}_{\alpha\times}\lambda^\alpha) = 1$

Restricted Quantification

- $\rightarrow$ provides *restricted universal quantification*
  - 'Every PA-theorem is true': $\forall x(\mathrm{Bew}_{PA}(x) \rightarrow Tx)$
- However, Field's $\rightarrow$ does not give

EASY $\forall x\phi(x) \rightarrow \forall(x)(\phi(x) \rightarrow \psi(x))$

HARD $\exists x(x = t) \wedge \phi(t) \models \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow \psi(t)$

Stronger, but not Quite Łukasiewicz

- Łukasiewicz $\aleph_0$-valued logic gives both (EASY) and (HARD)...
- ...but is inconsistent with naive truth.
- Field conjectures (slide 23):

  U $((\phi \rightarrow \psi) \rightarrow \psi) \rightarrow (\phi \vee \psi)$

  $Ł_{\aleph_0}$ − U, if consistent, 'probably ... the best possible naive truth theory'

- Can't we get any closer?

BCK Logic

- In the 1950s, C.A. Meredith proposed

  B $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$

  C $\phi \to ((\phi \to \psi) \to \psi)$ = (H) *as a law*

  K $\phi \to (\psi \to \phi)$ = Weakening = (E)

- BCK validates (EASY) and (HARD) *out of the box*.
- (HARD) even in the form of a law.
- BCK = $Ł_{\aleph_0} - U$ [Priest, 2008, 11.5.3-9]

BCK Algebras

- For $v(\phi \to \psi) = v(\phi) \Rightarrow v(\psi)$ BCK is sound and complete wrt algebras $\langle X, 1, \Rightarrow \rangle$ such that

Id If $x \Rightarrow y = 1$ and $y \Rightarrow x = 1$ then $x = y$

B_a $(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)) = 1$

C_a $x \Rightarrow ((x \Rightarrow y) \Rightarrow y) = 1$

K_a $x \Rightarrow (y \Rightarrow x) = 1$

- Partial order $x \leqslant y \ :\leftrightarrow \ x \Rightarrow y = 1$

Naive Truth with BCK?

- Field (slide 23):

  It's natural to hope that if we drop (U), we can get a naive truth theory in the weakened logic.

- Alas, for $\phi \vee \psi \;:\leftrightarrow\; \neg\phi \rightarrow \psi$ BCK proves $\phi \vee \neg\phi$.

Naive Truth with BCK Conditional?

- Nonetheless, BCK is a promising starting point: add $\rightarrow$ to paracomplete Kleene logic.
  - Naive set theory has been developed within BCK [Grišin, 1982, Bunder and daCosta, 1986]
- My suggestion:

  Let's build on the rich BCK literature to promote paracomplete truth theory.

A BCK-Kleene Algebra

- Bounded commutative BCK algebras are distributive lattices [Hoo, 2001].

$$\langle \{0, \mathfrak{u}, 1\}, 1, \Rightarrow \rangle$$

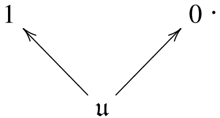
- Assume $0 \Rightarrow y = 1$ (0 is the lower bound) and

T $(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$

- $x^* = 0 \Rightarrow x, x \sqcup y = x^* \Rightarrow y$
- Let $v(\neg\phi) = v(\phi)^*$ and $v(\phi \vee \psi) = v(\phi) \sqcup v(\psi)$
- Do we get Strong Kleene logic for $\neg$ and $\vee$?

Kripke Fixed Point Model

- The existence of fixed point models follows from the Knaster-Tarski-theorem for coherent complete partially ordered sets ('ccpo').
- The Kleene value space $\langle\{0, \mathfrak{u}, 1\}, \leqslant_{\mathcal{S}}\rangle$ is a ccpo:



- The corresponding valuations $v : \mathcal{L}_{at} \mapsto \{0, \mathfrak{u}, 1\}$ with $v \leqslant_{\mathcal{V}} v'$ iff for every $\mathcal{L}_{at}$-sentence ϕ, $v(\phi) \leqslant_{\mathcal{S}} v'(\phi)$ make up a ccpo $\mathcal{V}$, too [Visser, 2004, lemma 7].
- Strong Kleene logic for $\mathcal{L}_{at} \backslash \mathcal{L}_a$-sentences provides a *monotone*[1] operator $K : \mathcal{V} \mapsto \mathcal{V}$

The Kripke Jump

- Let $m(\phi)$ be the value of the $\mathcal{L}_a$-sentence ϕ in the standard model.

K1 $K(v)(\phi) = m(\phi)$ for $\phi \in Sent_a$

K2 $K(v)(T\ulcorner\psi\urcorner) = v(\psi)$

K3 $K(v)(\neg\psi) = 1 - K(v)(\psi)$

K4 $K(v)(\psi \vee \chi) = max\{K(v)(\psi), K(v)(\chi)\}$

K5 $K(v)(\forall x\psi) = min\{K(v)(\psi(t/x)) | t \text{ closed term}\}$[2]

- There is a ccpo of fixed points $v_f = K(v_f)$.
- Especially, there's a least fixed point.

[2] Since we deal with arithmetic, it can be assumed that every object has a name in the language.

Adding the BCK Conditional

- It's still a BCK algebra.
- $v_f^+(\phi \to \psi) = v(\phi) \Rightarrow v(\psi)$
- $v_f^+(E) = v_f^+(\phi \to (\psi \to \phi)) = v(\phi) \Rightarrow (v(\psi) \to v(\phi)) \stackrel{K_a}{=} 1$
- $v_f^+(H_{law}) = v_f^+(\phi \to ((\phi \to \psi) \to \psi)) = (v(\phi) \Rightarrow ((v(\phi) \Rightarrow v(\psi)) \Rightarrow v(\psi))) \stackrel{C_a}{=} 1$

Summary

1. How does Field's new argument from rejection principles square with his probabilistic account of rejection?
2. Let's develop paracomplete truth with a BCK-conditional.

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